MACHINE BUILDING
МАШИНОСТРОЕНИЕ

Check for updates

UDC 62-192
https://doi.org/10.23947/2541-9129-2024-8-4-39-46

Original Theoretical Research

Fisher-Tippet Law Truncated Form for Loading Modeling of Machinery Structures

Anatoly A. Kotesov
Don State Technical University, Rostov-on-Don, Russian Federation
✉ a.kotesov@yandex.ru

EDN: OXXQDQ

Abstract

Introduction. Statistical data are used as a basis for assessing the reliability of engineering structures. However, incomplete data or inaccurate modeling of random variables may lead to an overestimation of reliability indicators. In practice, laws with infinitely decreasing or increasing distribution functions of an exponential family are usually used to model random variables characterizing the bearing capacity, load, and resource of engineering structures. To improve the accuracy of modeling of random variables, truncated forms of distribution laws are often used. These forms allow us to consider the random variable within a specified interval, excluding impossible values. Several studies have suggested using the Fisher-Tippett law with three parameters for modeling random variables related to the loading of engineering structures. The advantage of this law is that it limits the range of the random variable on the right side, but the left side of the distribution function decreases indefinitely, which is not ideal for load characteristics. To improve the accuracy of predicting random variables that characterize the load, it would be helpful to have a left-sided restriction using the Fisher-Tippett law. Currently, there are no descriptions of truncated forms of the distribution law in scientific literature. This article will explore the justification and development of a three-parameter truncated form of the Fisher-Tippett law and its use in calculation methods. The goal is to create a left-sided truncated version of the Fisher-Tippett distribution with three parameters to model random variables within a specific range.

Materials and Methods. The article provides a detailed description of the history of the Fisher-Tippet law, including its three-parameter form, and justifies the need for obtaining its truncated form.

Results. As a result of the research, a truncated form of the Fisher-Tippet three-parameter law in differential and integral forms was obtained and substantiated. The findings included graphs and calculations that demonstrated the normalization of a random variable within a given range.

Discussion and Conclusion. The conclusion was drawn about the advantages and disadvantages of the truncated form of the Fisher-Tippet law. The possibility of its practical application in the schematization of random loading processes under operating conditions and testing of machine elements and structures to assess fatigue life and determine fatigue resistance characteristics was established. The direction of further research is related to the practical use of the truncated form, particularly with the need to develop a method for evaluating the parameters of the truncated distribution and verifying the consistency of the proposed model.

Keywords: random variable, distribution law, truncated form, loading, reliability

Acknowledgements. The author would like to express his gratitude to the reviewers for their critical assessment of the submitted materials and their suggestions for improvement. These comments have contributed significantly to the enhancement of the quality of the presentation of the research results presented in this paper.

For citation. Kotesov AA. Fisher-Tippet Law Truncated Form for Loading Modeling of Machinery Structures. *Safety of Technogenic and Natural Systems.* 2024;8(4):39–46. https://doi.org/10.23947/2541-9129-2024-8-4-39-46

Оригинальное теоретическое исследование

Усеченная форма закона Фишера-Типпета для моделирования нагруженности машиностроительных конструкций

А.А. Котесов
Донской государственный технический университет, г. Ростов-на-Дону, Российская Федерация
✉ a.kotesov@yandex.ru

Аннотация
Введение. Статистические данные служат основой для оценки показателей надежности машиностроительных конструкций. Неполнота таких данных или неточность при моделировании случайных величин могут стать причиной завышенной оценки при определении показателей надежности. На практике для моделирования случайных величин, характеризующих несущую способность, нагруженность, ресурс машиностроительных конструкций, обычно применяют законы с бесконечно убывающими или возрастающими функциями распределения экспоненциального семейства. Для повышения точности при моделировании случайных величин часто используют усеченные формы законов распределения, которые позволяют рассматривать случайную величину в заданном интервале, исключая тем самым область невозможных значений. В ряде работ для моделирования случайных величин, характеризующих нагруженность машиностроительных конструкций, предлагается использовать закон Фишера-Типпета с тремя параметрами. Преимуществом данного закона является параметр, ограничивающий область определения рассматриваемой случайной величины справа, но при этом левая часть фунции распределения бесконечно убывает, что не совсем корректно для характеристик нагруженности. Поэтому для повышения точности моледирования случайных величин, характеризующих нагруженность, законом Фишера-Типпета целесообразно иметь ограничение слева. В настоящий момент в научной литературе не представлено описание усеченных форм для закона распределения, поэтому в предлагаемой статье будут рассмотрены обоснование и получение усеченной формы закона Фишера-Типпета с тремя параметрами и последующее использование ее в расчетных методиках. В связи с этим цель автора — получение левосторонней усеченной формы закона Фишера-Типпета с тремя параметрами для моделирования случайных величин в заданном интервале.
Материалы и методы. В статье подробно описана история получения, представлено описание и отличительные особенности закона Фишера-Типпета с тремя параметрами, а также обоснована необходимость получения его усеченной формы.
Результаты исследования. В результате исследования обоснована и получена усеченная форма закона Фишера-Типпета с тремя параметрами в дифференциальном и интегральном виде. Представлены результаты вычислений и графики функций, подтверждающие нормировку случайной величины в заданном интервале.
Обсуждение и заключение. Сделан вывод о преимуществах и недостатках усеченной формы закона Фишера-Типпета. Определена возможность практического применения усеченного закона при схематизации случайных процессов нагружения, возникающих в условиях эксплуатации или испытаний элементов машин и конструкций для оценки усталостной долговечности и определения характеристик сопротивления усталости. Направление дальнейших исследований связывается с практическим применением усеченной формы, в частности с необходимостью разработки методики для оценки параметров усеченного закона и проверки согласия предложенной модели.

Ключевые слова: случайная величина, закон распределения, усеченная форма, нагруженность, надежность

Благодарности. Автор выражает благодарность рецензентам, чья критическая оценка представленных материалов и предложения по их совершенствованию способствовали значительному повышению качества изложения результатов исследования, представленных в настоящей статье.

Для цитирования. Котесов А.А. Усеченная форма закона Фишера-Типпета для моделирования нагруженности машиностроительных конструкций. *Безопасность техногенных и природных систем.* 2024;8(4):39–46. https://doi.org/10.23947/2541-9129-2024-8-4-39-46

Introduction. Ensuring the reliability of mechanical engineering structures remains a crucial task at the present time. Failure of load-bearing elements in these structures during operation can lead to dangerous situations and economic losses. Consequently, the issues of determining reliability indicators, as well as the development of methods for a more accurate and reliable assessment of these indicators, as well as related research, are undoubtedly important and relevant.

In particular, the works of Professor V.E. Kas'yanov [1] emphasize that a reliable car is not necessarily expensive or more expensive than a less reliable one. Low reliability can be caused by various factors. One possible cause of sudden failures is the imperfection of calculation methods and the incompleteness of statistical data used to assess reliability indicators [2].

In practice, the laws of distribution of an exponential family with infinitely decreasing distribution functions are usually used to model random variables characterizing the bearing capacity, load, and resource of machine-building structures. This is described in detail in the works of V.V. Moskvichev and M.A. Kovalev [3], I.A. Panachev and I.V. Kuznetsov [4], G.Sh. Khazanovich and D.S. Apryshkin [5]. To increase accuracy, truncated forms of distribution laws are usually used, which allow us to consider a random variable in a given interval, focusing on available statistical data to determine the appropriate boundaries. V.E. Kas'yanov, L.P. Shchulkin [6], D.B. Demchenko [7], A.A. Kotesova [8] have proposed using the Fisher-Tippett law with three parameters for modeling random variables characterizing the loading of machine-building structures. The advantage of this law is the parameter that limits the area of definition of the random variable in question on the right, but at the same time the left part of the distribution function decreases infinitely, which is not entirely correct for load characteristics. Therefore, in order to increase the accuracy of modeling random variables characterizing the load, it is advisable to have a restriction on the left by the Fisher-Tippett law. Truncated forms for the most commonly applied laws are known and used in computational methods[1]. However, there is currently no description of a truncated Fisher-Tipet law in scientific literature. Therefore, this article addresses the issue of substantiating and obtaining a truncated form of the Fisher-Tippett law with three parameters and its subsequent use in calculation methods.

Materials and Methods. When modeling random variables, the Gaussian distribution (normal law) is most often used, and this is largely justified [9]. The limits of determining random variables in this model are set by the interval (–∞; ∞), which is not entirely correct for the characteristics of strength, load, and resources. Therefore, for modeling such random variables in the works of V.E. Kas'yanov, it is proposed to use the Weibull model with three parameters, which differs from the Gaussian model and the two-parameter Weibull in that it sets a slightly different interval for a random variable [c; ∞), where c is the shift parameter that determines the minimum value of a random variable, i.e. it has a restriction on the left.

The density function of the distribution of Weibull's law with three parameters defines the expression:

$$f(x \mid a,b,c) = \frac{b}{a}\left(\frac{x-c}{a}\right)^{b-1} e^{-\left(\frac{x-c}{a}\right)^{b}}, \tag{1}$$

where x — value of a random variable; a, b, c — parameters of scale, shape and shift of the distribution, respectively.

It is also proposed to use one of the three limiting forms of distributions attributed to type III by R. Fisher and K. Tippett to model the load characteristics. The distributions of Gumbel and Frechet were assigned to the first and second types, respectively [10].

The density function of the distribution of the third type defines the following expression:

$$f(x \mid k) = k(-x)^{k-1} e^{-(-x)^{k}}, \tag{2}$$

where x — value of a random variable; k — shape parameter.

It is obvious that this distribution is similar to the one-parameter Weibull distribution, which is also a special case of the generalized distribution of extreme values [11], only oriented to minimum values. By specifying the notation and adding additional parameters to expression (2) by analogy with the three-parameter Weibull's law, namely scale parameter — a and shift parameter (position) — c, we obtain the following expression for the distribution density function:

$$f(x \mid a,b,c) = \frac{b}{a}\left(\frac{c-x}{a}\right)^{b-1} e^{-\left(\frac{c-x}{a}\right)^{b}}, \tag{3}$$

where x — value of a random variable; a, b, c — the parameters of scale, shape and shift of the distribution, respectively.

Integrating expression (3) with respect to x, we obtain the distribution function of the law:

$$F(x) = \int f(x \mid a,b,c)\, dx,$$

$$F(x) = e^{-\left(\frac{c-x}{a}\right)^{b}}. \tag{4}$$

The resulting expression is proposed to be called the Fisher-Tippett law with three parameters, which, unlike the Weibull law, has a restriction on the right and defines the domain of determination of a random variable in the interval (–∞;c].

[1] RTM 24.090.25-76 *Lifting cranes. Calculation of the probability of failure-free operation of the elements.* (In Russ.) URL: https://gostrf.com/normadata/1/4293827/4293827795.htm (accessed: 15.05.2024).

As you can see, the Fisher-Tippett law with three parameters does not have a restriction on the left. The assumption of this model, given by the interval (–∞;0], may contradict the physical meaning of the random variables under consideration. In particular, the interval [0;c] will be more correct for load characteristics. Therefore, it is necessary to define the left-hand truncated form of the Fisher-Tippett law.

To solve this problem, it is proposed to use an approach similar to the approach considered in [12] and [13] to obtain a right-sided truncated Weibull law.

Research Results. To obtain the left-hand truncated form of the Fisher-Tippett law, it is necessary to set a condition under which all values of random variable x will be greater than a certain predetermined value t, which, in turn, will determine the truncation point of the law on the left, i.e. $x \in [t;c]$. Let us suppose that X_t ($t \geq 0$) denotes a left-truncated random variable distributed according to the Fisher–Tippett law, such that $F_t = c - x \mid c - x \geq t$, where $x \in W(a, b)$, a, $b > 0$ and $c > t$. Since $t \geq 0$, the value of the integral of the distribution density function f (x) in the interval [t;c] will be less than in the interval [–∞;c]:

$$\int_t^c f\left(x \mid a,b,c\right) dx < \int_{-\infty}^c f\left(x \mid a,b,c\right) dx = 1.$$

Therefore, to obtain a truncated form, it is necessary to redistribute the random variable in the interval [–∞;t] by determining the normalizing coefficient dependent on t:

$$K_{[t;c]} \cdot \int_t^c f\left(x \mid a,b,c\right) dx = \int_{-\infty}^c f\left(x \mid a,b,c\right) dx,$$

$$K_{[t;c]} \cdot \int_t^c f\left(x \mid a,b,c\right) dx = 1,$$

$$K_{[t;c]} = \frac{1}{\int_t^c f\left(x \mid a,b,c\right) dx},$$

$$K_{[t;c]} = \frac{1}{F(c) - F(t)},$$

where $F(c)$ and $F(t)$ — distribution function of the Fisher-Tippett law with three parameters, respectively, for $x = c$ and $x = t$.

Therefore, the truncated distribution function will be determined by the following expression:

$$f_{[t;c]}(x \mid a,b,c) = K_{[t;c]} \cdot f\left(x \mid a,b,c\right),$$

$$f_{[t;c]}(x \mid a,b,c) = \frac{f\left(x \mid a,b,c\right)}{F(x=c) - F(x=t)},$$

since $F(c) = 1$, we get:

$$f_{[t;c]}(x \mid a,b,c) = \frac{f\left(x \mid a,b,c\right)}{1 - F(t)}.$$

As a result, we obtain a distribution density function for the left-hand truncated form of the Fisher-Tippett law with three parameters in the range [t;c]:

$$f_{[t;c]}(x \mid a,b,c) = \frac{\frac{b}{a}\left(\frac{c-x}{a}\right)^{b-1} e^{-\left(\frac{c-x}{a}\right)^b}}{1 - e^{-\left(\frac{c-t}{a}\right)^b}}, \tag{5}$$

where x — value of a random variable; a, b, c — parameters of scale, shape and shift of the distribution, respectively; t — truncation point of the distribution (the minimum possible value of a random variable).

Obviously, taking $t = 0$, we get the following expression for the interval $[0;c]$:

$$f_{[0;c]}(x\,|\,a,b,c) = \frac{\dfrac{b}{a}\left(\dfrac{c-x}{a}\right)^{b-1} e^{-\left(\frac{c-x}{a}\right)^{b}}}{1-e^{-\left(\frac{c}{a}\right)^{b}}}. \tag{6}$$

In order to demonstrate the result obtained, the distribution parameters ($a = 10$; $b = 2.5$; $c = 15$) are arbitrarily set, graphs of the distribution density functions of initial (3) and truncated form (6) of the Fisher-Tippett law are constructed (Fig. 1). According to the specified parameters, certain integrals from the initial density function are calculated (3) and density functions of the truncated form (6) in the intervals $(-\infty;c]$, $(-\infty;0]$, $[0;c]$ are calculated. The calculations were performed in the Mathcad 14.0.0.163 software package with an acceptable convergence $1 \cdot 10^{-5}$. Table 1 presents the calculation results.

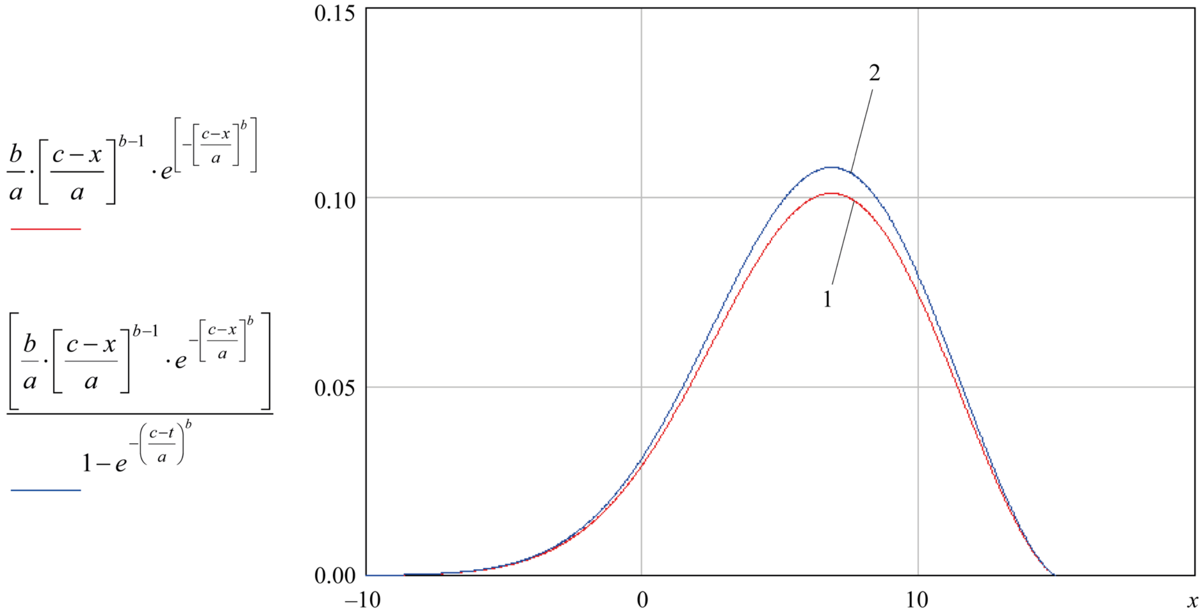

Fig. 1. Graphs of density distribution of the Fisher-Tippett law with three parameters ($a = 10$; $b = 2.5$; $c = 15$): 1 — initial density function; 2 — density function of the truncated law

Table 1

The results of calculating certain integrals from the density functions of the initial and truncated forms of the Fisher-Tippett law

Interval of a random variable	Limits of integration		Integration function and variable	
	lower	upper	$f(x\,|\,a,b,c)\,dx$	$f_{[0;c]}(x\,|\,a,b,c)\,dx$
$(-\infty;c]$	$-\infty$	c	0.99999999997581260	1.06788096063614480
$(-\infty;0]$	$-\infty$	0	0.06356603700150229	0.06788096066197107
$[0;c]$	0	c	0.93643395954036040	0.99999999631045600

The graphs of the functions in Figure 1 and the results of the numerical solution presented in Table 1 allow us to conclude that the analytically obtained distribution density function for the left-hand truncated form of the Fisher-Tippett law with three parameters (6) is correct and normalizes a random variable in the range $[0;c]$:

$$\int_0^c f_{[0;c]}(x\,|\,a,b,c)\,dx = 1.$$

But at the same time, the function retains the definition area to the left of the truncation point, i.e. in the interval $(-\infty;0]$, which must be taken into account when using the truncated form of the law in calculation methods:

$$\int_{-\infty}^{0} f_{[0;c]}(x \mid a,b,c)\,dx + \int_{0}^{c} f_{[0;c]}(x \mid a,b,c)\,dx = \int_{-\infty}^{c} f_{[0;c]}(x \mid a,b,c)\,dx > 1.$$

To obtain the distribution function of the truncated Fisher-Tippett law, we integrate expression (5):

$$F_{[t;c]}(x) = \int f_{[t;c]}(x \mid a,b,c)\,dx,$$

$$F_{[t;c]}(x) = \frac{e^{-\left(\frac{c-x}{a}\right)^{b}}}{1-e^{-\left(\frac{c-t}{a}\right)^{b}}}. \qquad (7)$$

Graphs of initial (4) and truncated (7) distribution functions are shown in Figure 2.

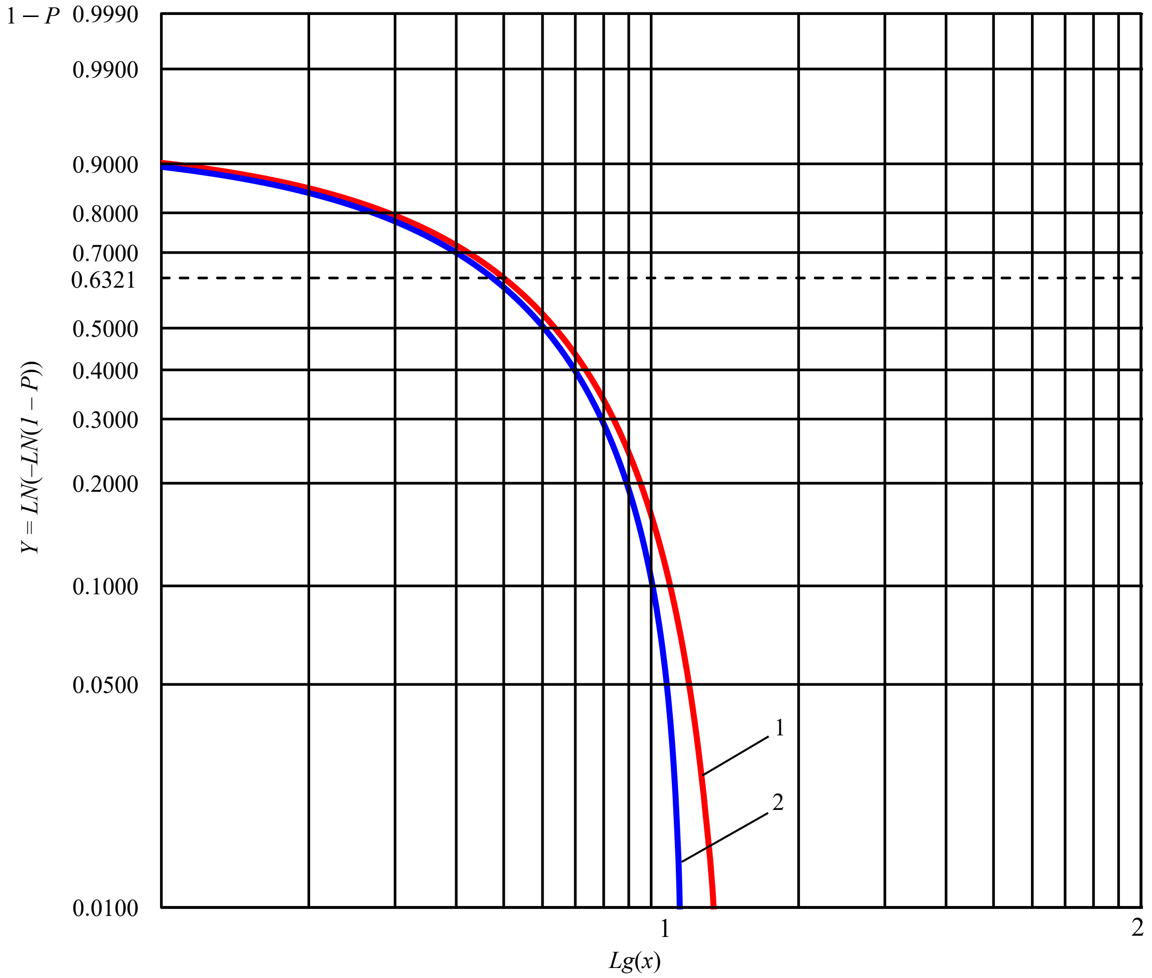

Fig. 2. Graphs of distribution functions of the Fisher-Tippett law with three parameters:
1 — original form; 2 — truncated form (t = 0); P — probability; x — random variable

From expression (7) we get the inverse function:

$$F^{-1}_{[t;c]}(x) = c - a\sqrt[b]{-ln\left(F_{[t;c]}(x)\cdot\left(1-e^{-\left(\frac{c-t}{a}\right)^{b}}\right)\right)}. \qquad (8)$$

Discussion and Conclusion. Thus, a left-sided truncated form of the Fisher-Tippett law with three parameters has been obtained, which can be used to schematize random loading processes that occur under operating conditions or tests of machine elements and structures according to GOST 25.101[2]. The use of the truncated form of the law makes it possible to

[2] GOST 25.101-83 *Strength calculation and testing. Representation of random loading of machine elements and structures and statistical evaluation of results.* (In Russ.) URL: https://docs.cntd.ru/document/1200012857 (accessed: 15.05.2024).

limit the interval of a random variable and exclude the area of impossible values to the left of the truncation point, which makes it possible to increase accuracy when using in computational methods for assessing the fatigue life of elements of mechanical engineering structures according to criteria for accumulation of fatigue damage, modeling the loading process during fatigue tests and calculating the characteristics of fatigue resistance. The calculation results shows that the distribution density function of the truncated law is correct and normalizes a random variable in a given interval, but at the same time retains the area of determination to the left of the truncation point, which is a disadvantage of the resulting model. Therefore, for the adequate application of the truncated law in calculation methods, it is necessary to introduce an appropriate restriction. The subject of future research is practical application of the truncated law, in particular, for statistical data processing, it is necessary to determine the methodology for estimating the parameters of the truncated law and determining confidence intervals. This includes obtaining expressions for estimating mathematical expectation and variance, as well as consider the possibility of using existing criteria of agreement.

References

1. Kasyanov VE. A Methodof Ensuringthe Absolute Reliability of Parts and Machines and Calculating the Increase of their Prices. *Engineering journal of Don*. 2016;1(40):19. (In Russ.)

2. Trukhanov VM. Prediction of the Life of Details, Units, Mechanisms, and the Devices in General at the Design Stage. *Journal of Machinery Manufacture and Reliability*. 2013;3:38–42. (In Russ.)

3. Moskvichev VV, Kovalev MA. Assessment of Operational Reliability Indicators of Mine Hydraulic Excavators. *Journal of Siberian Federal University. Engineering & Technologies.* 2020;13(6):745–756. (In Russ.) http://doi.org/10.17516/1999-494X-0263

4. Panachev IA, Kuznetsov IV. Substantiation of the Loading of Elements of Metal Structures of Heavy-Duty Dump Trucks during the Transportation of Rock Mass in the Kuzbass Sections. In: *Proceedings of the International Scientific and practical Conference "New Approaches to the Development of the Coal Industry".* Кемерово; 2013. P. 61–64. (In Russ.)

5. Khazanovich GS, Apryshkin DS. Assessment of Load of Load-Bearing Elements of the Passenger Elevator Based on Regular Monitoring Results. *Safety of Technogenic and Natural Systems.* 2020;(1):32–42. (In Russ.) http://doi.org/10.23947/2541-9129-2020-1-32-42

6. Kasyanov VE, Schulkin LP. Determination of the Maximum Loading of Parts with the Help of Modeling. *Science Review*. 2014;10(3):671–674. (In Russ.)

7. Demchenko DB, Kasyanov VE. Optimization Method for Static Calculation of Construction Designs with the Use of Probabilistic Laws with Restrictions. *Engineering journal of Don*. 2013;2(25):84. (In Russ.)

8. Kotesov AA, Kasyanov VE, Kotesova AA. Model for Ensuring the Reliability of Metal Structures of Lifting Cranes during Their Service Period. *Vestnik Rostovskogo Gosudarstvennogo Universiteta Putey Soobshcheniya.* 2020;4(80):30–39. (In Russ.) http://doi.org/10.46973/0201-727X_2020_4_30

9. Horrace WC. Moments of the Truncated Normal Distribution. *Journal of Productivity Analysis.* 2015;43:133–138. https://doi.org/10.1007/s11123-013-0381-8

10. Fisher RA, Tippet LHC. Limiting Forms of the Frequency Distribution of the Longest of Smallest Member of Sample. *Mathematical Proceedings of the Cambridge Philosophical Society.* 1928;24(2),180–190. https://doi.org/10.1017/S0305004100015681

11. Bashir Ahmed Albashir Abdulali, Mohd Aftar Abu Bakar, Kamarulzaman Ibrahim, Noratiqah Mohd Ariff. Extreme Value Distributions: An Overview of Estimation and Simulation. *Journal of Probability and Statistics.* 2022:5449751. https://doi.org/10.1155/2022/5449751

12. Salman Abbas, Muhammad Farooq, Jumanah Ahmed Darwish, Saman Hanif Shahbaz, Muhammad Qaiser Shahbaz. Truncated Weibull–Exponential Distribution: Methods and Applications. *Scientific Reports.* 2023;13:20849. https://doi.org/10.1038/s41598-023-48288-x

13. Crénin F. Truncated Weibull Distribution Functions and Moments. *Journal of Productivity Analysis.* 2015;43:133–138. http://dx.doi.org/10.2139/ssrn.2690255

About the Author:

Anatoly A. Kotesov, Cand. Sci. (Eng.), Associate Professor of the Department of Operation of Transport Systems and Logistics, Don State Technical University (1, Gagarin Sq., Rostov-on-Don, 344003, Russian Federation), SPIN-code, ORCID, ScopusID, ResearcherID, a.kotesov@yandex.ru

Conflict of Interest Statement: **the author does not have any conflict of interest.**

The author has read and approved the final version of manuscript.

Об авторе:

Анатолий Анатольевич Котесов, кандидат технических наук, доцент кафедры эксплуатации транспортных систем и логистики Донского государственного технического университета (344003, Российская Федерация, г. Ростов-на-Дону, пл. Гагарина, 1), SPIN-код, ORCID, ScopusID, ResearcherID, a.kotesov@yandex.ru

***Конфликт интересов*: автор заявляет об отсутствии конфликта интересов.**

Автор прочитал и одобрил окончательный вариант рукописи.

Received / Поступила в редакцию 14.08.2024
Revised / Поступила после рецензирования 10.09.2024
Accepted / Принята к публикации 17.09.2024